JOURNAL OF COMPUTATIONAL PHYSICS 13, 439–444 (1973)

Note

Partial Implicitization

The steady-state solution of a simplified form of the governing equations of fluid mechanics has been obtained for a wide variety of flows; however, the steady-state solution to the full Navier–Stokes equations for complicated flows is generally much more difficult to obtain. Currently solutions to the Navier–Stokes equations are generally obtained through some form of time marching to the steady-state. Examples of currently used techniques can be found in Refs. [1–4]. The time-dependent form of these equations takes a relatively large number of time steps to reach steady state for the maximum time step is restricted by a stability limit. With a number of different time-dependent methods available, a way of judging the relative merits of a particular method is to use Burgers equation (see [5]) as a model of the Navier–Stokes equations (see Refs. [3, 4, 6, and 7]), for Burgers equation is simpler yet similar in structure to the Navier–Stokes equations. In finite-difference form, it is easier to resolve the stability, accuracy, and convergence rate of Burgers equation than the full Navier–Stokes equations.

In Ref. [3], three frequently used techniques were tested using Burgers equation; however, in all three methods, the maximum time step is limited by their corresponding stability limit. If the stability criterion were to be relaxed, then the solution could proceed to the steady state in fewer steps, thus speeding convergence.

In the following paragraphs, Burgers equation will be used as a model to test an explicit numerical technique which will be shown to be unconditionally stable. The method to be developed below is a one-step explicit technique resulting from a partial implicitization of the difference equation. The stability analysis will show that the technique is unconditionally stable and numerical tests will show the technique to be accurate.

The method will be demonstrated through the use of Burgers equation,

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = \nu \frac{\partial^2 U}{\partial x^2}. \tag{1}$$

This equation was introduced by Burgers [5] in an attempt to develop a model of free turbulence; however, in the present application, Eq. (1) will be used to model a diffuse shock wave through the application of the following boundary conditions:

$$U(x, t) = 1, \qquad x \to -\infty,$$
$$U(x, t) = 0, \qquad x \to +\infty.$$

Since only the steady-state solution is desired, the following wave oriented transformation is applied to Eq. (1):

$$\eta = x - \bar{U}t,$$
$$\hat{t} = t,$$

where $\bar{U}$ is the steady-state wave speed. Equation (1) becomes

$$\frac{\partial U}{\partial \hat{t}} + U_0 \frac{\partial U}{\partial \eta} = \nu \frac{\partial^2 U}{\partial \eta^2}, \tag{2}$$

where $U_0 = U - \bar{U}$ and the wave speed $\bar{U} = \frac{1}{2}$.
The boundary conditions under the transformations become,

$$U(\eta, \hat{t}) = 1 \quad \text{for} \quad \eta \to -\infty,$$
$$U(\eta, \hat{t}) = 0 \quad \text{for} \quad \eta \to +\infty.$$

The steady-state solution to Eq. (2) subject to the above boundary conditions is

$$U(\eta) = \tfrac{1}{2}(1 - \tanh(\eta/4\nu)). \tag{3}$$

Equation (2) can be expressed in finite-difference form using a backward difference on the time term and central differences on the spatial terms. The spatial differences are written at the advanced time as in a fully implicit procedure

$$U_j^{N+1} - U_j^{N} + r_j(U_{j+1}^{N+1} - U_{j-1}^{N+1}) - S(U_{j+1}^{N+1} - 2U_j^{N+1} + U_{j-1}^{N+1}) = 0, \tag{4}$$

where

$$r_j = U_{0j}\, \Delta\hat{t}/2\Delta\eta,$$
$$S = \nu\, \Delta\hat{t}/\Delta\eta^2.$$

Equation (4) reduces to

$$-(r_j + S)\, U_{j-1}^{N+1} + (1 + 2S)\, U_j^{N+1} - (S - r_j)\, U_{j+1}^{N+1} = U_j^{N}. \tag{5}$$

Equation (4) is written at points $J - I$, J, and $J + I$; however, the points $J - 2$ and $J + 2$ are considered explicitly. The system of three simultaneous equations is of the form

Point $J - 1$

$$(1 + 2S)\, U_{j-1}^{N+1} - (S - r_{j-1})\, U_j^{N+1} = U_{j-1}^{N} + (r_{j-1} + S)\, U_{j-2}^{N},$$

Point J

$$-(r_j + S)\, U_{j-1}^{N+1} + (1 + 2S)\, U_j^{N+1} - (S - r_j)\, U_{j+1}^{N+1} = U_j^N,$$

Point $J + 1$

$$-(r_{j+1} + S)\, U_j^{N+1} + (1 + 2S)\, U_{j+1}^{N+1} = U_{j+1}^N + (S - r_{j+1})\, U_{j+2}^N\,,$$

To obtain the solution at point J we use Cramer's rule (see Ref. [8]) which gives an equation which will be used at all interior points in the finite-difference mesh except for the two points immediately adjacent to the boundaries. At these points, the above system is solved to obtain equations for U_{j-1}^{N+1} and U_{j+1}^{N+1}. Note that the equations for U_{j-1}^{N+1} and U_{j+1}^{N+1} are used only for the two points adjacent to the boundary.

The solution using Cramer's rule is obtained from

$$U_j^{N+1} = \frac{\begin{vmatrix} (1 + 2S) & (U_{j-1}^N + (r_{j-1} + S)\, U_{j-2}^N) & 0 \\ -(r_j + S) & U_j^N & -(S - r_j) \\ 0 & (U_{j+1}^N + (S - r_{j+1})\, U_{j+2}^N) & (1 + 2S) \end{vmatrix}}{\begin{vmatrix} (1 + 2S) & -(S - r_{j-1}) & 0 \\ -(r_j + S) & (1 + 2S) & -(S - r_j) \\ 0 & -(r_{j+1} + S) & (1 + 2S) \end{vmatrix}}$$

The result is

$$U_j^{N+1} = \bar{D}((1 + 2S)\, U_j^N + (S - r_j)\, U_{j+1}^N + (S - r_j)(S - r_{j+1})\, U_{j+2}^N + (r_j + S)\, U_{j-1}^N + (r_j + S)(r_{j-1} + S)\, U_{j-2}^N\,, \tag{6}$$

where

$$\bar{D} = \frac{1}{(1 + 2S)^2 - \{(S - r_j)(r_{j+1} + S) + (S - r_{j-1})(r_j + S)\}}\,. \tag{6a}$$

Equation (6) now involves five points from the previous time level only, hence, Eq. (6) is an explicit equation obtained from partial implicilization of the difference form of the governing equation.

The stability analysis of Eq. (6) will be performed using a linearized von Neumann analysis. Following Ref. [4], a finite Fourier series expansion to Eq. (6) will be made with the Fourier components of the form

$$U_j^N = V^N e^{iK_\eta(J\Delta\eta)}. \tag{7}$$

Defining the phase angle as $\theta = K_\eta \Delta\eta$, Eq. (7) becomes

$$U_j^N = V^N e^{ij\theta}. \tag{8}$$

Substituting the appropriate forms of Eq. (8) into Eq. (6) results in

$$V^{N+1} = V^N \bar{D}\{(1 + 2S) + 2S\cos\theta + 2S^2\cos 2\theta + 2r^2\cos 2\theta - 2irS\sin\theta - 4irS\sin 2\theta\}. \tag{9}$$

Looking first at Eq. (6a)

$$\bar{D} = 1/(1 + 4S + 2S^2 + 2r^2). \tag{9a}$$

Since $0 \leqslant S < +\infty$ and $-\infty < r < +\infty$ then the denominator of $\bar{D}$, Eq. (9a), is always greater than zero, and, hence, no singularity exists in Eq. (9).

Defining $G = V^{N+1}/V^N$ Eq. (9) becomes

$$G = \bar{D}\{1 + 2S + 2S\cos\theta + 2S^2\cos 2\theta + 2r^2\cos 2\theta - 2ir\sin\theta - 4irS\sin 2\theta\}. \tag{10}$$

The von Neumann condition for stability requires

$$|G| \leqslant 1 \tag{11}$$

with the result that Eq. (10) becomes

$$\begin{aligned} |G| = \bar{D}\{&8S^2r^2\cos^2 2\theta + (1 + 2S)^2 + 4S(1 + 2S)\cos\theta \\ &+ 4S^2(1 + 2S)\cos 2\theta + 4r^2(1 + 2S)\cos 2\theta \\ &+ 4S^2\cos^2\theta + 8S^3\cos\theta\cos 2\theta + 4S^4\cos^2 2\theta \\ &+ 8Sr^2\cos\theta\cos 2\theta + 4r^4\cos^2 2\theta \\ &+ 4r^2\sin^2\theta + 16r^2S\sin\theta\sin 2\theta \\ &+ 16r^2S^2\sin^2 2\theta\}^{1/2}. \end{aligned} \tag{12}$$

Since Eq. (12) is rather complicated, first consider the simpler limits of ($S = 0$, $r \neq 0$) and ($S \neq 0$, $r = 0$).

For ($S = 0$, $r \neq 0$) Eq. (12) becomes

$$|G| = \frac{(1 + 4r^2\cos^2\theta + 4r^4\cos^2 2\theta)^{1/2}}{1 + 2r^2}. \tag{13}$$

The maximum value of Eq. (13) occurs at $\theta = 0$ giving

$$|G| = \frac{(1 + 4r^2 + 4r^4)^{1/2}}{1 + 2r^2} = 1.$$

Thus, Eq. (12) satisfies the von Neumann stability criterion in the limit ($S = 0$, $r \neq 0$). Similarly it can be shown when ($S \neq 0$, $r = 0$)

$$|G| = \frac{(1 + 8S + 20S^2 + 16S^3 + 4S^4)^{1/2}}{1 + 4S + 2S^2} = 1.$$

Equation (12) satisfies the von Neumann stability criterion in both sets of limits, implying Eq. (6) is stable for all $\Delta\hat{t}$ in the limit conditions. By numerical evaluation it was determined that the maximum of Eq. (12) occurs at $\theta = 0$ giving

$$|G| = \frac{\{1 + 8S + 20S^2 + 16S^3 + 4S^4 + 16Sr^2 + 8S^2r^2 + 4r^2 + 4r^4\}^{1/2}}{1 + 4S + 2S^2 + 2r^2},$$

$$|G| = 1.$$

Thus, Eq. (12) satisfies the stability criterion for all $\Delta\hat{t}$, and, hence, Eq. (6) is an unconditionally stable solution for Eq. (1).

Equation (6) and the two additional relations for U_{j+1}^{N+1} and U_{j-1}^{N+1} (to be used adjacent to the boundaries) were programmed for numerical solution over the range $-5 \leqslant \eta \leqslant 5$ with $\nu = 1/8$ and $\Delta\eta = 0.1$. Additionally the simple explicit form of Eq. (1) was programmed using the same range and initial conditions. The explicit solution was run at 90 % of the stability limit. A comparison of the results is given in the following table. The partially implicit result was obtained with $\Delta\hat{t} = 36$ (900 times the explicit limit).

η	Exact	Partially implicit	Explicit
−0.7	0.94267	0.94471	0.94471
−0.6	0.91683	0.91929	0.91929
−0.5	0.88079	0.88363	0.88363
−0.4	0.83202	0.83505	0.83505
−0.3	0.76852	0.77143	0.77143
−0.2	0.68997	0.69231	0.69231
−0.1	0.59868	0.60000	0.60000
0	0.50000	0.50000	0.50000
0.1	0.40131	0.40000	0.40000
0.2	0.31003	0.30769	0.30769
0.3	0.23147	0.22857	0.22857
0.4	0.16798	0.16495	0.16495
0.5	0.11920	0.11636	0.11636
0.6	0.08317	0.08071	0.08071
0.7	0.05324	0.05529	0.05529

The partially implicit results are to five decimal places identical to the explicit results indicating that in the partial implicitization process no additional truncation errors of any significance have been introduced. The explicit solution took 549 time steps while the partially implicit result took only 147 to reach steady state.

The use of partial implicitization, while demonstrated for Burgers equation, is not limited only to this particular case but can be applied in principle to similar problems, including the Navier–Stokes equations.

References

1. J. S. Allen and S. I. Cheng, Numerical solutions of the compressible Navier–Stokes equations for the laminar near wake, *Phys. Fluids* **13** (1970), 37.
2. P. J. Roache and T. J. Mueller, Numerical Solutions of Compressible and Incompressible Laminar Separated Flows, AIAA Paper 68-741, June 1968.
3. J. E. Carter, Numerical Solutions of the Supersonic, Laminar Flow Over a Two-Dimensional Compression Corner, Virginia Polytechnic Institute and State University Ph.D. Thesis, August 1971.
4. P. J. Roache, "Computational Fluid Dynamics," Hermosa Publishers, Albuquerque, NM, 1972.
5. J. Burgers, A Mathematical Model Illustrating the Theory of Turbulence, *Adv. Appl. Mech.* **1** (1948), 171.
6. S. I. Cheng, Accuracy of difference formulation of Navier–Stokes equations: High-speed computing in Fluid dynamics, *Phys. Fluids Suppl.* **II** (1969).
7. D. S. Watanabe and J. R. Flood, An Implicit Fourth Order Difference Method for Viscous Flows, Coordinated Science Laboratory, Report R-572, University of Illinois, Urbana, Illinois, June 1972.
8. I. S. Sokolnikoff and R. M. Redheffer, "Mathematics of Physics and Modern Engineering," McGraw-Hill, New York, 1958.

Received: February 5, 1973

Randolph A. Graves, Jr.

Gas Radiation Section
Space Systems Division
NASA-Langley Research Center
Hampton, Virginia 23365